UNIT 14

QUADRATIC FUNCTIONS

Real Numbers:

Rationals: Any terminating or repeating decimal is rational. It can be written in the form $\frac{a}{b}$, $b \neq 0$ (i.e., 3.6 or .213213).

Irrationals: Any non-terminating, non-repeating decimal (i.e., π or .010203...). Irrationals cannot be written in the form $\frac{a}{b}$, $b \neq 0$.

Rational numbers are sometimes used to approximate irrational numbers.

From Decimal to Rational Form:

If $N = .54\overline{54}$, then

$$\begin{aligned} 100N &= 54.\overline{54} \\ -\quad N &= \ \ .\overline{54} \\ \hline 99N &= 54.00 \end{aligned}$$

- *Multiply by 10^n; n = number of repeating digits, (2).*
- *Subtract the number from 100 times the number.*
- *Solve for N.*

$$N = \frac{54}{99}$$

$$= \frac{6}{11}$$

- *$\frac{6}{11}$ is the rational form for $.54\overline{54}$.*

Square Root:

The *square roots* of 9 are 3 and -3 since $3^2 = 9$ and $(-3)^2 = 9$.

The *principal* square root of 9 is $\sqrt{9} = 3$ (the *positive* square root).

The *negative* square root of 9 is $-\sqrt{9} = -3$.

The square root of a negative number is *not* a real number, (i.e., $\sqrt{-9} \neq 3$ since $3^2 = 9$; and $\sqrt{-9} \neq -3$ since $(-3)^2 = 9$).

A perfect square is a non-negative number whose principal square root is a rational number, (i.e., 49 is a perfect square since $\sqrt{49} = 7$; 1.44 is a perfect square since $\sqrt{1.44} = 1.2$; $\frac{16}{25}$ is a perfect square since $\sqrt{\frac{16}{25}} = \frac{4}{5}$.

Simplifying Square Roots:

1. $\sqrt{4 \cdot 25} = \sqrt{4}\ \sqrt{25} = 2 \cdot 5 = 10$ — • $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$.

2. $\sqrt{5} \cdot \sqrt{5} = 5$ — • $\sqrt{a} \cdot \sqrt{a} = a$.

3. $\sqrt{54} = \sqrt{9 \cdot 6} = \sqrt{9}\ \sqrt{6} = 3\sqrt{6}$ — • *Find the greatest perfect square factor and take its square root.*

4. $11\sqrt{72} = 11\sqrt{36 \cdot 2} = 11 \cdot 6\sqrt{2} = 66\sqrt{2}$ — • *Multiply coefficients of a radical.*

5. $\sqrt{x^6} = x^3$ — • $\sqrt{x^n} = x^{n/2}$ *(n = an even integer, $x \geq 0$).*

6. $\sqrt{m^4y^{10}p^6} = \sqrt{m^4}\ \sqrt{y^{10}}\ \sqrt{p^6} = m^2y^5p^3$ — • *Use rules for Ex. 1 & 5 .*

7. $\sqrt{32x^8y^{12}} = \sqrt{16x^8y^{12}}\ \sqrt{2} = 4x^4y^6\ \sqrt{2}$ — • $\sqrt{32} = \sqrt{16}\ \sqrt{2}$; *extract roots of the perfect squares: 16, x^3 and y^{12}.*

8. $\sqrt{x^7} = \sqrt{x^6 \cdot x} = \sqrt{x^6}\ \sqrt{x} = x^3\sqrt{x}$ — • *Rewrite $\sqrt{x^n}$, where "n" is an odd exponent, as $\sqrt{x^{n-1} \cdot x}$, where (n-1) is even.*

9. $-3x^2y\ \sqrt{24x^9y^4} = -3x^2y\ \sqrt{4x^8y^4 \cdot 6x}$ — • *Use rules for Ex. 3 & 8.*
   $= -3x^2y\ \sqrt{4x^8y^4} \cdot \sqrt{6x}$ — • *Ex. 1 rule above.*
   $= (-3x^2y)(2x^4y^2)\ \sqrt{6x}$ — • *Ex. 1 & 5 rules above.*
   $= -6x^6y^3\ \sqrt{6x}$ — • *Multiply like factors.*

Adding and Subtracting Radicals:

1. $3\sqrt{5} + 7\sqrt{5} = (3 + 7)\sqrt{5} = 10\sqrt{5}$ — • *Use Distributive Property when adding terms with the same radical factors.*

2. $5\sqrt{2} + 4\sqrt{50} = 5\sqrt{2} + 4\sqrt{25}\ \sqrt{2}$ — • *Simplify each radical first.*
   $= 5\sqrt{2} + 20\sqrt{2}$ — • $4\sqrt{25}\ \sqrt{2} = 4 \cdot 5\ \sqrt{2} = 20\sqrt{2}$.
   $= 25\sqrt{2}$ — • *Combine like radicals.*

3. $y\sqrt{25x^3y} + 3x\sqrt{xy^3} - \sqrt{81x^3y^3} =$
   $5xy\sqrt{xy} + 3xy\sqrt{xy} - 9xy\sqrt{xy} =$ — • *Simplify each radical.*
   $-xy\sqrt{xy}$ — • *Combine like radicals.*

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Multiplying and Dividing Radicals:

1. $3\sqrt{3} \cdot -2\sqrt{7} = -6\sqrt{21}$

   • *Multiply coefficients; then multiply radicands.*

2. $2\sqrt{3}(4\sqrt{5} - 3\sqrt{2}) = 8\sqrt{15} - 6\sqrt{6}$

   • *Distribute $2\sqrt{3}$ and simplify each product.*

3. $(2\sqrt{5} + 3\sqrt{2})(3\sqrt{5} - 5\sqrt{2}) =$
   $6 \cdot 5 - 10\sqrt{10} + 9\sqrt{10} - 15 \cdot 2 =$
   $30 - \sqrt{10} - 30 =$
   $-\sqrt{10}$

   • *Use FOIL method and multiply as two binomials. Simplify each product and combine like terms and like radicals.*

4. $\dfrac{\sqrt{128x^4y^6}}{\sqrt{2x^2y^2}} = \sqrt{\dfrac{128x^4y^6}{2x^2y^2}}$
   $= \sqrt{64x^2y^4}$
   $= 8xy^2$

   • $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$
   • *Simplify the quotient.*
   • *Simplify the radical.*

5. $\dfrac{3\sqrt{8xy^7}}{\sqrt{18x^5y^3}} = \dfrac{3\sqrt{8xy^7} \cdot \sqrt{2xy}}{\sqrt{18x^5y^3} \cdot \sqrt{2xy}}$
   $= \dfrac{3\sqrt{16x^2y^8}}{\sqrt{36x^6y^4}}$

   • *Multiply numerator and denominator by the factors necessary to make the denominator radicand a perfect square. (This is called "rationalizing the denominator.")*

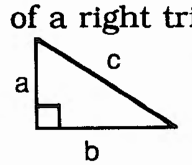
   $= \dfrac{12xy^4}{6x^3y^2} = \dfrac{2y^2}{x^2}$

   • *Then simplify each expression.*

Pythagorean Theorem:

The legs, *a* and *b*, and the hypotenuse, *c*, of a right triangle have the relationship expressed by the equation:

$$a^2 + b^2 = c^2$$



If any two dimensions are known, the third can be found.

Find the hypotenuse of a right triangle with legs of 8cm and 6cm:

$c^2 = a^2 + b^2$
$c^2 = 8^2 + 6^2$
$c^2 = 64 + 36$
$c^2 = 100$
$c = \sqrt{100} = 10$

Thus, the hypotenuse is 10cm long.

Lesson 14.0

Review Problems

Tell if each number is rational or irrational: (If rational, change it to $\frac{a}{b}$ form.)

1. $.7\overline{7}$
2. $.313233...$
3. $2.1415926...$
4. $3.\overline{142}$
5. $0.\overline{325}$
6. $.4\overline{9}$

Simplify each radical expression:

7. $\sqrt{45}$
8. $\sqrt{288}$
9. $\sqrt{72x^4}$
10. $\sqrt{32x^8y^{12}}$
11. $\sqrt{24xy^5}$
12. $\sqrt{8x^3y^8z^9}$

Find each square root correct to three decimal places:

13. $\sqrt{78}$
14. $\sqrt{288}$
15. $\sqrt{39}$

Simplify each radical expression:

16. $\sqrt{25 \cdot 81}$
17. $-3\sqrt{369}$
18. $\sqrt{12x^6y^{10}}$
19. $-\sqrt{121x^8y^{12}z^{16}}$
20. $\sqrt{27b^9}$
21. $2x^2y\sqrt{60x^3y^8}$
22. $\sqrt{(a+b)^3}$
23. $5\sqrt{2} + 7\sqrt{2}$
24. $2\sqrt{ab^3} - 3b\sqrt{25ab}$
25. $3\sqrt{2}\,(\sqrt{3} - 4\sqrt{2})$
26. $(2\sqrt{6} + 3\sqrt{10})(3\sqrt{6} - 5\sqrt{10})$
27. $(\sqrt{12} - \sqrt{3})^2$
28. $\dfrac{2}{\sqrt{5}}$
29. $\dfrac{24m^2n^3}{\sqrt{20m^7n^4}}$

Solve:

30. If a right triangle has one leg of 9 in. and a hypotenuse of 13 in., find the length of the second leg. Use simple radical form for answer.

Rule: A *radical equation* is one which contains one or more radicals (square roots). To solve such an equation:

1. Transform it into one equation with a single radical term for one of its sides.
2. Square both sides.
3. Repeat steps 1 and 2 if a radical still appears.
4. Solve the resulting (radical-free) equation.
5. Check the solution(s) in the original equation.

Example: 1. Solve:

$\sqrt{x} = 3$ • *Single radical on left.*

$(\sqrt{x})^2 = 3^2$ • *Square both sides.*

$x = 9$ • *No radical remains. Is 9 a solution?*

Check:

$\sqrt{x} = 3$ • *Try 9 as a solution.*

$\sqrt{9} = 3$ • *Substitute 9 for "x."*

$3 = 3$ • *True; the solution is 9.*

Rules: Squaring both sides of an equation does not always give an equivalent equation.

Extraneous solutions are those which do not satisfy the original equation.

Example: 2. $x = 2$ • *2 is the only solution.*

$x^2 = 2^2$ • *Square both sides.*

$x^2 = 4$ • *$2^2 = 4$; also $(-2)^2 = 4$.*

• *So, both 2 and -2 are solutions.*

Since -2 does not satisfy the original equation $x = 2$, -2 is an *extraneous solution.*

***Note:* Always check for extraneous solutions when solving radical equations.**

Example: 3. Solve: $\sqrt{3x + 1} + 6 = 2$

$\sqrt{3x + 1}$	$= -4$	• *Radical term is alone.*
$(\sqrt{3x + 1})^2$	$= (-4)^2$	• *Square each side.*
$3x + 1$	$= 16$	• *$(\sqrt{a})^2 = a$.*
$3x$	$= 15$	• *Add -1.*
x	$= 5$	• *Check this solution.*

Check:

$\sqrt{3x + 1} + 6$	$= 2$	• *Use original equation.*
$\sqrt{3(5) + 1} + 6$	$= 2$	• *Substitute 5 for "x."*
$\sqrt{15 + 1} + 6$	$= 2$	
$\sqrt{16} + 6$	$= 2$	
$4 + 6$	$= 2$	
10	$= 2$	• *False.*

So, 5 is an extraneous solution. $\sqrt{3x + 1} + 6 = 2$ has no solution in the set of real numbers.

Rule: Squaring can result in a quadratic equation.

Examples: 4. Solve:

$x - 2$	$= \sqrt{19 - 6x}$	• *Radical is alone.*
$(x - 2)^2$	$= (\sqrt{19 - 6x})^2$	• *Square each side.*
$x^2 - 4x + 4$	$= 19 - 6x$	• *Quadratic equation results.*
$x^2 + 2x - 15$	$= 0$	• *Quadratic in standard form.*
$(x + 5)(x - 3)$	$= 0$	• *Factor.*
$x + 5 = 0$ *or* $x - 3 = 0$		• *Let each factor equal 0.*
$x = -5$ *or* $x = 3$		• *Solve resulting equations.*

Check:

$x - 2 = \sqrt{19 - 6x}$	$x - 2 = \sqrt{19 - 6x}$
$-5 - 2 = \sqrt{19 - 6(-5)}$ • *Try -5*	$3 - 2 = \sqrt{19 - 6(3)}$ • *Try 3*
$-7 = \sqrt{19 + 30}$	$1 = \sqrt{19 - 18}$
$-7 = \sqrt{49}$	$1 = \sqrt{1}$
$-7 = 7$ • *False, -5 is an extraneous solution.*	$1 = 1$ • *True, 3 is the solution.*

Example: 5. Solve:

$\sqrt{x} - 2 = \sqrt{x + 27} - 5$ • *Radical on each side.*

$\sqrt{x} + 3 = \sqrt{x + 27}$ • *Add 5; $\sqrt{x + 27}$ is now alone.*

$(\sqrt{x} + 3)^2 = (\sqrt{x + 27})^2$ • *Square each side.*

$x + 6\sqrt{x} + 9 = x + 27$ • *Single radical remains.*

$6\sqrt{x} = 18$ • *Add -x and -9 to each side.*

$\sqrt{x} = 3$ • *Divide by 6.*

$(\sqrt{x})^2 = 3^2$ • *Square each side again.*

$x = 9$ • *Check this solution.*

Check:

$\sqrt{x} - 2 = \sqrt{x + 27} - 5$ • *Use original equation.*

$\sqrt{9} - 2 = \sqrt{9 + 27} - 5$ • *Substitute 9 for "x."*

$3 - 2 = \sqrt{36} - 5$ • *Simplify.*

$1 = 6 - 5$

$1 = 1$ • *True; so 9 is a solution.*

HOMEWORK

Solve:

1. $\sqrt{x} = 5$
2. $\sqrt{y} = 4$
3. $\sqrt{4x} = 6$
4. $\sqrt{x} + 3 = 4$
5. $\sqrt{y + 3} = 3$
6. $\sqrt{2x + 2} = \sqrt{x + 4}$
7. $\sqrt{5x + 4} = \sqrt{3x + 10}$
8. $\sqrt{4y + 4} + 10 = 2$
9. $\sqrt{2x + 5} = \sqrt{3x}$
10. $\sqrt{9x + 6} = \sqrt{6x + 9}$
11. $\sqrt{3x + 1} - 5 = -6$
12. $\sqrt{2p + 6} = 4$
13. $\sqrt{\frac{1}{2}x + 2} = \sqrt{x - 5}$

Solve:

14. Two times the square root of a number is 4. Find the number.
15. Five times the square root of a number is 15. Find the number.
16. A number is increased by 3. If the square root is multiplied by 3, the result is 15. Find the number.

Solve:

17. $\sqrt{2x + 2} = \sqrt{x + 20}$
18. $\sqrt{k - 2} = \sqrt{2k - 28}$
19. $\sqrt{5y + 3} = \sqrt{y - 9}$
20. $\sqrt{4a - 3} - 2 = \sqrt{2a - 5}$

Rule: An equation of the form $x^2 = a$ $(a \geq 0)$ may be solved by the factor method or by *extraction of roots.*

Example: 1. Solve $x^2 = 9$:

Method A (Factor Method)	Method B (Extraction Method)
$x^2 = 9$	$x^2 = 9$
$x^2 - 9 = 0$	$x = \sqrt{9}$ *or* $x = -\sqrt{9}$ • *Extract roots.*
$(x + 3)(x - 3) = 0$ • *Factor.*	$x = 3$ *or* $x = -3$
$x + 3 = 0$ *or* $x - 3 = 0$	
$x = -3$ *or* $x = 3$	
	• *Method B is faster.*

Rule: If $x^2 = a$ $(a \geq 0)$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$.

Example: 2. Solve:

$x^2 = 50$

$x = \sqrt{50}$ *or* $x = -\sqrt{50}$ • *Extract roots on each side.*

$x = 5\sqrt{2}$ *or* $x = -5\sqrt{2}$ • $\sqrt{50} - \sqrt{25}\sqrt{2} = 5\sqrt{2}.$

So, the solutions are $5\sqrt{2}$ and $-5\sqrt{2}$. These solutions are written more simply as $\pm 5\sqrt{2}$.

Rule: When one member of an equation is a perfect square containing the variable, such as $x^2 = c$, or $(kx)^2 = c$, or $(kx + b)^2 = c$, the equation may be solved by extraction of roots.

Example: 3. Solve: $(3x + 4)^2 = 49$

$3x + 4 = \sqrt{49}$ *or* $3x + 4 = -\sqrt{49}$ • *Extract roots.*

$3x + 4 = 7$ *or* $3x + 4 = -7$

$3x = 3$ *or* $3x = -11$

$x = 1$ $\quad x = \frac{-11}{3}$ • *Check these solutions.*

Check:

• *Try 1*	• *Try $\frac{-11}{3}$*
$(3x + 4)^2 = 49$	$(3x + 4)^2 = 49$
$(3 \cdot 1 + 4)^2 = 49$	$(3 \cdot \frac{-11}{3} + 4)^2 = 49$
$(3 + 4)^2 = 49$	$(-11 + 4)^2 = 49$
$7^2 = 49$	$(-7)^2 = 49$
$49 = 49$ TRUE	$49 = 49$ TRUE

Since both are true, 1 and $\frac{-11}{3}$ are the solutions.

Practice: Solve: A. $(x + 4)^2 = 64$ B. $(2x + 2)^2 = 36$

Rule: Solutions often need to be rationalized or otherwise simplified.

Example: 4. Solve:

$5x^2 = 3$

$x^2 = \frac{3}{5}$ • *Divide by 5 to get a perfect square member.*

$x = \sqrt{\frac{3}{5}}$ *or* $x = -\sqrt{\frac{3}{5}}$ • *Then extract roots and rationalize.*

$x = \frac{\sqrt{15}}{5}$ *or* $x = -\frac{\sqrt{15}}{5}$ • $\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3} \cdot \sqrt{5}}{\sqrt{5}\,\sqrt{5}} = \frac{\sqrt{15}}{5}$.

So, the solutions are $\pm\frac{\sqrt{15}}{5}$.

Lesson 14.2

Example: 5. Solve:

$(2x - 8)^2 = 28$

$2x - 8 = \sqrt{28}$ *or* $2x - 8 = -\sqrt{28}$ • *Extract roots.*

$2x = 8 + \sqrt{28}$ *or* $2x = 8 - \sqrt{28}$ • *Add 8 to each side.*

$2x = 8 + 2\sqrt{7}$ *or* $2x = 8 - 2\sqrt{7}$ • $\sqrt{28} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$.

$x = 4 + \sqrt{7}$ *or* $x = 4 - \sqrt{7}$ • *Divide by 2.*

The solutions are $4 \pm\sqrt{7}$.

Practice: Solve: C. $2x^2 = 6$ D. $(4x - 20)^2 = 32$

Example: 6. Solve:

$5(4x + 2)^2 - 6 = 94$ • *Get* $(4x + 2)^2$ *alone.*

$5(4x + 2)^2 = 100$ • *Add 6 to each side.*

$(4x + 2)^2 = 20$ • *Divide by 5.*

$4x + 2 = \sqrt{20}$ *or* $4x + 2 = -\sqrt{20}$ • *Extract roots.*

$4x = -2 + \sqrt{20}$ *or* $4x = -2 - \sqrt{20}$ • *Add -2.*

$4x = -2 + 2\sqrt{5}$ *or* $4x = -2 - 2\sqrt{5}$ • $\sqrt{20} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$.

$x = \dfrac{-1 + \sqrt{5}}{2}$ *or* $x = \dfrac{-1 - \sqrt{5}}{2}$ • *Divide by 4 and simplify.*

So, the solutions are $\dfrac{1 \pm \sqrt{5}}{2}$.

HOMEWORK

Solve:

1. $x^2 = 13$
2. $x^2 = 7$
3. $k^2 = 8$
4. $t^2 = 18$
5. $y^2 = 48$
6. $a^2 = 54$
7. $(x + 3)^2 = 9$
8. $(y - 2)^2 = 16$
9. $(m + 4)^2 = 64$
10. $(s - 5)^2 = 36$
11. $(2k + 8)^2 = 144$
12. $4x^2 = 5$
13. $3f^2 = 7$
14. $3n^2 = 36$
15. $5x^2 = 125$
16. $10y^2 = 200$
17. $(x - 5)^2 = 10$
18. $(p + 2)^2 = 27$
19. $(q - 3)^2 = 28$
20. $(d - 7)^2 = 24$
21. $(y + 4)^2 = 36$
22. $(x - 9)^2 = 8$
23. $5(2x - 1)^2 = 20$
24. $3(3x - 3)^2 + 3 = 33$
25. $2(5x - 7)^2 - 2 = 28$
26. $9(4x + 3)^2 - 7 = 65$
27. $4(6x - 2)^2 + 5 = 53$
28. $2(2x + 2)^2 + 6 = 38$

Solve:

29. If 8 is added to a number, the square of the result is 25. Find the number.

30. The square of three less than a number is 9. Find the number.

Lesson 14.3
Solving Equations by Completing the Square — Unit 14

Rule: Squaring a binomial results in a trinomial which is called a *perfect square trinomial.*

Example:

1. $(x + 3)^2 = (x + 3)(x + 3) = x^2 + 6x + 9$ (a perfect square trinomial)

   $(3x + 2)^2 = (3x + 2)(3x + 2) = 9x^2 + 12x + 4$ (a perfect square trinomial)

Rule: To square a binomial, $(x + b)^2$:

1. Square the first term for the first term of the trinomial: x^2.
2. Find the product of the two terms (bx) and double it for the middle term of the trinomial: $2(bx)$.
3. Square the last term for the last term of the trinomial: b^2.
4. $(x + b)^2 = x^2 + 2bx + b^2$

Examples:

2. $(x + 5)^2 = x^2 + 2(5x) + 5^2 = x^2 + 10x + 25$

   5x is doubled — twice 5 — 5 squared

3. $(x - 4)^2 = x^2 + 2(-4x) + (-4)^2 = x^2 - 8x + 16$

   -4x is doubled — twice -4 — -4 squared

Rules: Conversely, to produce a perfect square trinomial, given an expression $x^2 + 2bx$, add the square of half the linear term coefficient:

$$\left[\frac{1}{2}(2bx)\right]^2 = b^2$$

$x^2 + 2bx + b^2$ is a perfect square trinomial. This technique is called *completing the square* and may be used to solve quadratic equations.

Examples: 4. Make a perfect square trinomial by adding a constant:

$x^2 - 6x + ?$

half of -6 = -3 — • *Half of the linear term coefficient = -3.*

$(-3)^2 = 9$ — • *$b^2 = 9$.*

$x^2 - 6x + 9$ — • *Add 9 to the expression.*

$x^2 - 6x + 9 = (x - 3)^2$ — • *This is a perfect square.*

5. Make a perfect square trinomial by adding a constant:

$x^2 + 7x + ?$

half of 7 = $\frac{7}{2}$ — • *Half of the linear term coefficient = $\frac{7}{2}$.*

$\left(\frac{7}{2}\right)^2 = \frac{49}{4}$ — • *$b^2 = \frac{49}{4}$.*

$x^2 + 7x + \frac{49}{4}$ — • *$\left(\frac{7}{2}\right)^2$ must be added.*

$x^2 + 7x + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$ — • *This is a perfect square.*

6. Solve by completing the square:

$x^2 + 12x = -11$

$x^2 + 12x + ? = -11 + ?$ — • *First, complete the square.*

$x^2 + 12x + 36 = -11 + 36$ — • *Add $\left(\frac{12}{2}\right)^2 = 6^2 = 36$ to each side.*

$(x + 6)^2 = 25$ — • *Factor the left side.*

$x + 6 = \sqrt{25}$ *or* $x + 6 = -\sqrt{25}$ — • *Extract roots.*

$x + 6 = 5$ *or* $x + 6 = 5$

$x = -1$ *or* $x = -11$ — • *Check these solutions.*

The solutions are -1 and -11.

Lesson 14.3

Example: 7. Solve by completing the square:

$x^2 - 10x + 4 = 0$

$x^2 - 10x = -4$ • *Get constant on right side.*

$x^2 - 10x + 25 = -4 + 25$ • *Add* $\left(\frac{-10}{2}\right)^2 = (-5)^2 = 25$ *to each side.*

$(x - 5)^2 = 21$ • *Factor the perfect square.*

$x - 5 = \sqrt{21}$ *or* $x - 5 = -\sqrt{21}$

$x = 5 + \sqrt{21}$ *or* $x = 5 - \sqrt{21}$ • *Add 5 to each side.*

The solutions are $5 \pm \sqrt{21}$.

Rule: To solve a quadratic equation of the form $ax^2 + bx + c = 0$, with $a \neq 0$ or 1, first divide both sides by *a* to obtain an x^2 coefficient of *1*. Then complete the square and extract roots. Fractions often occur.

Example: 8. Solve by completing the square:

$3x^2 - 2x - 8 = 0$

$x^2 - \frac{2}{3}x - \frac{8}{3} = 0$ • *Divide by 3 to obtain* $1x^2$.

$x^2 - \frac{2}{3}x = \frac{8}{3}$ • *Add* $\frac{8}{3}$ *to get constant on right.*

$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{8}{3} + \frac{1}{9}$ • *Add* $\left(\frac{-2}{3} \cdot \frac{1}{2}\right)^2 = \left(\frac{-2}{6}\right)^2 = \left(\frac{-1}{3}\right)^2 = \frac{1}{9}$. *This is the square of half the "x" coefficient.*

$\left(x - \frac{1}{3}\right)^2 = \frac{25}{9}$ • *Factor the perfect square.*

$x - \frac{1}{3} = \sqrt{\frac{25}{9}}$ *or* $x - \frac{1}{3} = -\sqrt{\frac{25}{9}}$ • *Extract roots on both sides.*

$x - \frac{1}{3} = \frac{5}{3}$ *or* $x - \frac{1}{3} = -\frac{5}{3}$

$x = 2$ *or* $x = \frac{-4}{3}$ • $\frac{5}{3} + \frac{1}{3} = \frac{6}{3} = 2$; $\frac{-5}{3} + \frac{1}{3} = \frac{-4}{3}$.

The solutions are 2 and $\frac{-4}{3}$.

HOMEWORK

Solve by completing the square:

1. $x^2 + 4x = 5$
2. $x^2 + 4x = 12$
3. $x^2 - 6x = 16$
4. $x^2 - 2x = 15$
5. $n^2 - 10n = -21$
6. $x^2 + 2x = 8$
7. $x^2 - 9x = -20$
8. $b^2 - 4b + 4 = 0$
9. $y^2 + 4y = 12$
10. $s^2 + 6s = 91$
11. $p^2 + 4p = 32$
12. $m^2 + 2m - 6 = 0$
13. $y^2 + 4y + 2 = 0$
14. $a^2 + 14a + 6 = 0$
15. $x^2 + 5x - 14 = 0$
16. $x^2 - 7x - 18 = 0$
17. $t^2 - 5t + 4 = 0$
18. $b^2 + 3b - 108 = 0$
19. $3x^2 + 2x - 1 = 0$
20. $2d^2 - 7d = 4$

Rule: The *quadratic formula* is obtained by completing the square to solve the general quadratic equation $ax^2 + bx + c = 0$:

$ax^2 + bx + c = 0$	• *General quadratic equation.*
$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$	• *Divide by "a" to get $1x^2$.*
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	• *Add $-\frac{c}{a}$ to get constant on right.*
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$	• $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right)^2$*, the square of half the "x" coefficient.*
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$	• *LCD is $4a^2$; $\frac{c}{a} = \frac{4ac}{4a^2}$.*
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	• *Factor left; combine right side.*
$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$ or $x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$	• *Extract roots.*
$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$	• *Simplify right side.*
$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$	• *Add $\frac{-b}{2a}$.*
$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$	• *Simplify.*
So, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	• *Quadratic Formula.*

Rule: To solve *any* quadratic equation:

1. Write the equation in the form $ax^2 + bx + c = 0$ where $a > 0$.
2. Substitute *a*, *b* and *c* from the above equation into the *quadratic formula*:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
3. Simplify the results. No real solution exists if $(b^2 - 4ac) < 0$.

Examples: 1. Solve:

$x^2 - 5x + 6 = 0$

$1x^2 - 5x + 6 = 0$ • *Determine a, b, and c: a = 1; b = -5; c = 6.*

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ • *Use quadratic formula.*

$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2a}$ • *Substitute for "a," "b" and "c."*

$x = \frac{5 \pm \sqrt{1}}{2}$ • *Simplify.*

$x = \frac{5 + 1}{2}$ *or* $x = \frac{5 - 1}{2}$ • $\sqrt{1} = 1$.

$x = 3$ *or* $x = 2$ • *Solution.*

2. Solve:

$-2 = -2x - x^2$

$x^2 + 2x - 2 = 0$ • *Rewrite in standard form: a = 1; b = 2; c = -2.*

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ • *Use quadratic formula.*

$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)}$ • *Substitute for "a," "b" and "c."*

$x = \frac{-2 \pm \sqrt{12}}{2}$

$x = \frac{-2 \pm 2\sqrt{3}}{2}$ • $\sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$.

$x = \frac{2(-1 \pm \sqrt{3})}{2}$ • *Factor out 2 and reduce.*

$x = -1 \pm \sqrt{3}$ • *Two irrational roots.*

Lesson 14.4

Example: 3. Solve:

$-x^2 = 4 + 2x$

$x^2 + 2x + 4 = 0$ • *Rewrite: a = 1; b = 2; c = 4.*

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ • *Use formula.*

$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$ • *Substitute for "a," "b" and "c."*

$x = \frac{-2 \pm \sqrt{-12}}{2}$

Since $\sqrt{-12}$ is not a real number, there is no real solution.

Rule: When an equation has fractional coefficients or constants, it is usually easier to solve if fractions are "cleared." Multiply each side by the LCD to eliminate fractions.

Example: 4. Solve:

$\frac{1}{6}x^2 + \frac{5}{6}x = \frac{1}{2}$

$6\left(\frac{1}{6}x^2 + \frac{5}{6}x\right) = \left(\frac{1}{2}\right) \cdot 6$ • *Multiply by LCD, 6.*

$x^2 + 5x = 3$

$x^2 + 5x - 3 = 0$ • *Put in standard form.*

$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(-3)}}{2(1)}$ • *Use quadratic formula: a = 1; b = 5; c = -3.*

$x = \frac{-5 \pm \sqrt{37}}{2}$ • *Solution.*

Example: 5. Solve: $\frac{2}{3}x - \frac{1}{15}x^2 = \frac{5}{3}$

$15\left(\frac{2}{3}x - \frac{1}{15}x^2\right) = \left(\frac{5}{3}\right)15$ • *LCD is 15.*

$10x - x^2 = 25$ • *Multiply by 15.*

$x^2 - 10x + 25 = 0$ • *Put in standard form.*

$x = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(25)}}{2(1)}$ • *Use formula: a = 1; b = -10; c = 25.*

$x = \frac{10 \pm \sqrt{0}}{2} = 5$ • *The solution is 5.*

***NOTE:* A quadratic equation may have two rational roots (example 1), two irrational roots (examples 2 and 4), no real roots (example 3), or one rational root (example 5).**

HOMEWORK

Solve by using the quadratic formula:

1. $x^2 - 3x + 2 = 0$
2. $x^2 + 3x - 10 = 0$
3. $x^2 - 7x - 18 = 0$
4. $x^2 - 4x - 21 = 0$
5. $x^2 - 7x + 9 = 0$
6. $x^2 + 2x - 5 = 0$
7. $x^2 + 5x + 2 = 0$
8. $x^2 + 3x - 1 = 0$
9. $x^2 = -4x + 6$
10. $x^2 - 3x - 5 = 0$
11. $2x^2 - 4x + 2 = 0$
12. $x^2 - 1 = x$
13. $3x^2 - 2x = 1$
14. $-9x^2 + 6x - 1 = 0$
15. $2x^2 - 7x + 3 = 0$
16. $\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{2} = 0$
17. $\frac{1}{2}x^2 + \frac{2}{3}x + 1 = 0$
18. $\frac{3}{x - 1} = 1 + \frac{x - 4}{x - 3}$
19. $\frac{x}{x + 3} - 2 = \frac{x + 3}{x - 2}$

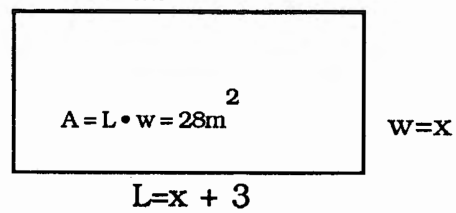
Solve:

20. For what value of "K" will there be no real solution for $x^2 + 4x + K = 0$?

Rule: Many area problems involve solving quadratic equations which have two roots. If so, only a *positive* root may be a solution. All negative values are rejected since measurements (such as length and width) can never be negative. Always check reasonable roots.

Example: 1. A cement slab is 3 meters longer than it is wide. If its total area is 28 square meters, find its dimensions.

Represent each dimension. • *Draw a sketch.*



The length is expressed in terms of the width, so let the variable be the width.

Width = x Length = x + 3	• *Length is 3 meters longer than width.*
$A = L \cdot w$	• *Use area formula.*
$28 = (x + 3)x$	• *Substitute known information.*
$28 = x^2 + 3x$	• *Write the equation.*
$0 = x^2 + 3x - 28$	• *Standard quadratic form.*
$0 = (x + 7)(x - 4)$	• *Factor.*
$x + 7 = 0$ *or* $x - 4 = 0$	• *Let each factor equal 0.*
$x = -7$ *or* $x = 4$	• *Two roots. Reject -7. (Distance can never be negative.)*

Width = x = 4
Length = x + 3 = 4 + 3 = 7

$A = L \cdot w$ $28 = 7 \cdot 4$ $28 = 28$, TRUE	• *Check.*

So, the dimensions are 4 meters by 7 meters.

Rule: The quadratic formula is often used to solve area problems when the quadratic equation does not factor.

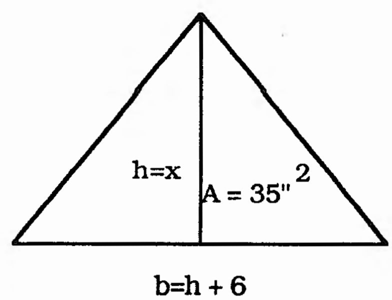
Note: **The formula for the area of a triangle is:**

$$A = \frac{1}{2}bh \quad \textit{or} \quad A = \frac{bh}{2}$$

Example: 2. Find the length (to the nearest tenth) of the base and height of a triangle if the base is 6 inches longer than the height and the area is 35 square inches.

Represent the dimensions. • *Draw a sketch.*

Since the base is given in terms of the height, let the variable be the height.



Height = x

Base = x + 6 • *Base is 6" longer than height.*

$A = \frac{1}{2}bh$ • *Use area formula.*

$35 = \frac{1}{2}(x + 6)(x)$ • *Substitute known information.*

$35 = \frac{1}{2}(x^2 + 6x)$ • *Distribute the "x."*

$70 = x^2 + 6x$ • *Multiply both sides by 2.*

$0 = x^2 + 6x - 70$ • *Standard form (won't factor).*

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(-70)}}{2(1)}$$

- *Use quadratic formula: $a = 1$, $b = 6$; $c = -70$.*

$$x = \frac{-6 \pm \sqrt{316}}{2}$$

- *Simplify radicand.*

$$x = \frac{-6 \pm 2\sqrt{79}}{2}$$

- $\sqrt{316} = \sqrt{4 \cdot 79} = \sqrt{4}\sqrt{79} = 2\sqrt{79}$.

$$x = -3 \pm \sqrt{79}$$

- *Reject $-3-\sqrt{79}$, since it is negative*

$$x = -3 + \sqrt{79}$$

- *Answer in radical form. (Distance can never be negative.)*

$$x = -3 + 8.9$$

- *Find $\sqrt{79} = 8.888$ in table or on calculator. Round to 8.9.*

Height = x = 5.9

Base = x + 6 = 5.9 + 6 = 11.9

So, the base is 11.9" and the height is 5.9" (to the nearest tenth).

HOMEWORK

Solve: (Give answers to the nearest tenth.)

1. The length of a rectangle is 2cm more than the width. The area is $8cm^2$. Find the length and width.

2. The length of a rectangle is 2cm less than twice the width. The area is $40cm^2$. Find the length and width.

3. The length of a rectangle is 1m more than four times the width. The area is $18m^2$. Find the length and width.

4. The length of a rectangle is 1cm more than twice the width. The area is $55cm^2$. Find the length and width.

5. The base of a triangle is 3m less than the height. The area is $5m^2$. Find the base and height.

6. The base of a triangle is 1cm less than three times the height. The area is $35cm^2$. Find the base and height.

7. The length of a rectangle is 3m more than the width. The area is $26m^2$. Find the length and width.

8. The length of a rectangle is 3cm less than twice the width. The area is $45cm^2$. Find the length and width.

9. The base of a triangle is 1cm less than three times the height. The area is $13cm^2$. Find the base and height.

10. The perimeter of a rectangle is 16cm. The area is $12cm^2$. Find the length and width.

Rule: A quadratic equation of the form $y = ax^2 + bx + c$, $a \neq 0$, has a curve called a *parabola* for its graph. The curve has symmetry through a vertical line, $x = k$, called the *axis of symmetry*.

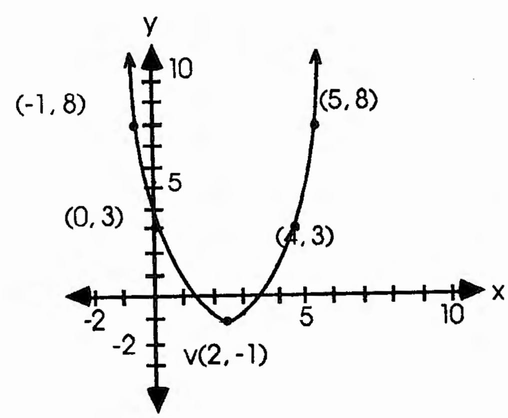
A table of values may be used to obtain points on the graph of such a function.

Select *x-values* and substitute them into the equation to find the corresponding *y-value* of each point. The lowest (or highest) point on the curve is called the turning point or *vertex*. It is always on the axis of symmetry and is also halfway between the x-intercepts.

Example: 1. Graph the function $y = x^2 - 4x + 3$:

Table of Values

x	x^2	-	4x	+	3	y	Point on the curve
-1	$(-1)^2$	-	4(-1)	+	3	8	(-1, 8)
0	0^2	-	4(0)	+	3	3	(0,3)
1	1^2	-	4(1)	+	3	0	(1,0)
2	2^2	-	4(2)	+	3	-1	(2, -1)
3	3^2	-	4(3)	+	3	0	(3,0)
4	4^2	-	4(4)	+	3	3	(4,3)
5	5^2	-	4(5)	+	3	8	(5,8)



x-intercepts: 1 and 3
Axis of symmetry: x = 2
Vertex: (2, -1)

Example: 2. Find the vertex algebraically for the function:

$$y = x^2 - 4x + 3$$

A. First, let y be 0 and solve for x. This gives the x-intercepts of the function:

$0 = x^2 - 4x + 3$
$0 = (x - 1)(x - 3)$
$x - 1 = 0$ *or* $x - 3 = 0$
$x = 1$ *or* $x = 3$

x-intercepts are 1 and 3.

B. Axis of symmetry, $x = k$, is halfway between the x-intercepts, 1 and 3:

$$x = \frac{1 + 3}{2} = \frac{4}{2} = 2$$

$x = 2$ is the axis of symmetry and also the x-coordinate for the turning point.

C. Substitute 2 for x in the original equation, $y = x^2 - 4x + 3$, to find the y-coordinate of the turning point:

$y = 2^2 - 4(2) + 3$
$y = 4 - 8 + 3$
$y = -1$

D. The vertex is (2, -1).

Rule: The vertex of a parabola, $y = ax^2 + bx + c$, $a \neq 0$, has x-coordinate:

$$x = \frac{-b}{2a}$$

Lesson 14.6

Examples: 3. Find the vertex for $y = x^2 - 4x + 3$ (again):

$x = \frac{-b}{2a}$ • *Vertex of a parabola.*

$x = \frac{-(-4)}{2(1)}$ • *From standard form: $a = 1, b = -4, c = 3$.*

$x = \frac{4}{2}$

$x = 2$ • *x-coordinate.*

$y = x^2 - 4x + 3$

$y = (2)^2 - 4(2) + 3$ • *Substitute for "x."*

$y = 4 - 8 + 3$

$y = -1$ • *y-coordinate.*

Vertex is at (2, -1).

4. Given the function $y = -2x^2 - 12x - 10$, show:

a) y-intercept
b) x-intercepts
c) axis of symmetry
d) vertex
e) graph

$$y = -2x^2 - 12x - 10$$

a) $y = -2x^2 - 12x - 10$
$y = -2(0)^2 - 12(0) - 10$ • *Let $x = 0$.*
$y = -10$ • *Solve for "y."*
y-intercept is at (0, -10).

b) $y = -2x^2 - 12x - 10$
$0 = -2x^2 - 12x - 10$ • *Let $y = 0$. Solve for "x."*
$0 = (2x + 2)(-x - 5)$ • *Factor.*

$2x + 2 = 0$ *or* $-x - 5 = 0$
$2x = -2$ *or* $-x = 5$
$x = -1$ *or* $x = -5$

x-intercepts are -1 and -5.

c) $x = \frac{(-1) + (-5)}{2} = \frac{-6}{2} = -3$

Axis of symmetry is at x = -3.

• *Axis of symmetry is halfway between x-intercepts, -1 and -5.*

d) $x = \frac{-b}{2a}$

• *Vertex of parabola.*

$x = \frac{-(-12)}{2(-2)}$

• *From standard form: a = -2, b = -12, c = -10.*

$x = \frac{12}{-4} = -3$

• *x-coordinate.*

$y = -2x^2 - 12x - 10$

$y = -2(-3)^2 - 12(-3) - 10$

• *Substitute for "x."*

$y = -2(9) - 12(-3) - 10$

$y = -18 + 36 - 10$

$y = 8$

• *y-coordinate.*

Vertex is at (-3, 8)

e)

• *Plot points found in a) through d).*

• *Plot additional points to fill in graph.*

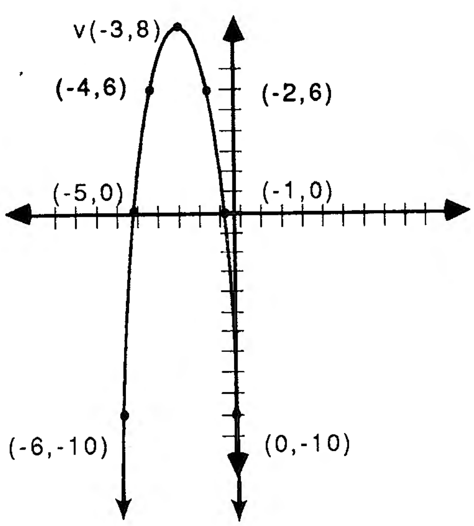


Table of Values

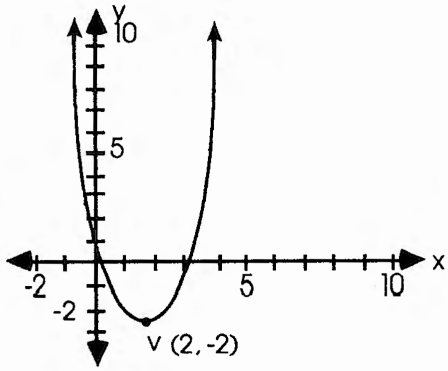
x	$-2x^2$	- 12x	- 10	y	Point on curve
-2	$-2(-2)^2$	-12(-2)	-10	6	(-2,6)
-4	$-2(-4)^2$	-12(-4)	-10	6	(-4,6)
-6	$-2(-6)^2$	-12(-6)	-10	-10	(-6,-10)

Lesson 14.6

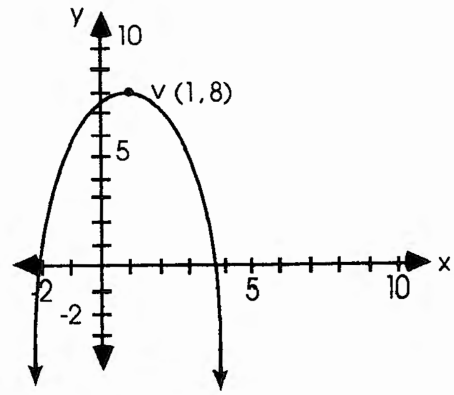
Rule: For the quadratic function $y = ax^2 + bx + c$; $a \neq 0$, the graph opens upward with a *minimum* value at the vertex when a is positive, and it opens downward with a *maximum* value when a is negative.

Examples:

5. $y = 2x^2 - 8x + 6$
x^2 coefficient is positive. Graph opens upward and the turning point is a minimum. The minimum function value is -2.



6. $y = -3x^2 + 6x + 5$
x^2 coefficient is negative. Graph opens downward and the turning point is a maximum. The maximum function value is 8.



HOMEWORK

Determine the coordinates of the turning point of each parabola:

1. $y = x^2 - 4x + 3$
2. $y = x^2 + 2x - 3$
3. $y = x^2 - 9$

Draw the graph for each of the following and label the turning point:

4. $y = x^2 + 6x + 8$
5. $y = -x^2 + 4x - 3$
6. $y = -2x^2 + 8x - 5$
7. $y = x^2 - 4x + 6$
8. $y = 2x^2 + x - 3$
9. Find the minimum value of y for $y = x^2 - 2x + 1$.
10. Find the maximum value of y for $y = -5x^2 - 10x + 3$.